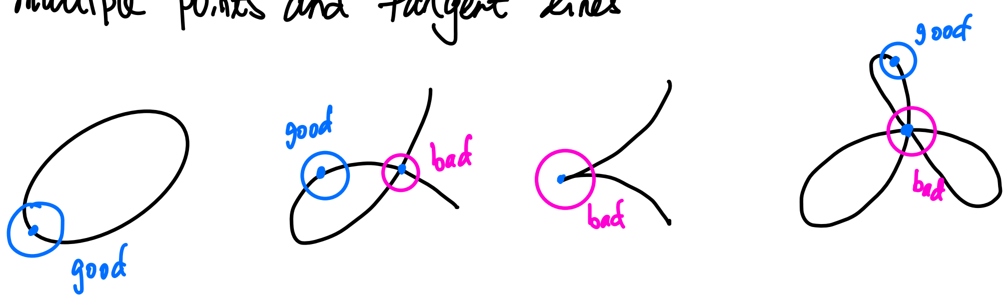
§3 local properties of plane curves

§3.1. multiple points and tangent lines



Fact: $F, G \in k[X,Y]$ without multiple factors.

$$V(F) = V(G) \overset{\S 1.6}{\Leftrightarrow} \exists \lambda \in k^* \text{ s.t. } F = \lambda G$$

$\Downarrow$ 推了

- $F, G \in k[X,Y]$ are equivalent if $F = \lambda G$ for some $\lambda \in k^*$.
- affine plane curve := equivalent class of nonconst. poly.

  e.g. the plane curve $Y^2 - X^3$. or $Y^2 = X^3$.
- degree of a curve := deg. of a definition poly. for the curve.

  Line = deg. one curve.

$$F = \prod \underset{\substack{\uparrow \\ \text{irr}}}{F_i^{e_i}} \Rightarrow \begin{cases} F_i = \text{component of } F \\ e_i = \text{multiplicity of } F_i \end{cases}$$

$e_i = 1 \Rightarrow F_i$ = simple component of F

$e_i > 1 \Rightarrow F_i$ = multiple component of F

$V(F) = V(\prod F_i) \Rightarrow$ recover $V(F_i)$ from $V(F)$

$F = \text{irr.} \Rightarrow V(F) = \text{variety}$, denote

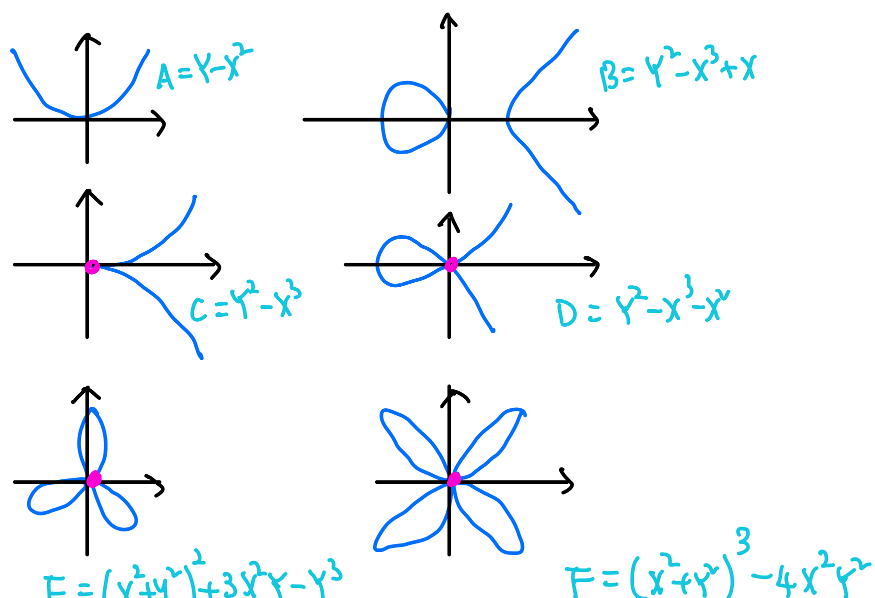
$\Gamma(F) := \Gamma(V(F))$, $k(F) = k(V(F))$, $\mathcal{O}_P(F) = \mathcal{O}_P(V(F))$

(非标准记号)

$F = $ curve, $P = (a,b) \in F$ (i.e. $F(a,b) = 0$)

局部最重要的几何量：切线.
怎么从代数方法求得？

Example: (real parts of) some curve in $\mathbb{A}^2(\mathbb{C})$.



$P = $ simple point of F $\overset{\text{def}}{\Longleftrightarrow}$ $(F_x(P), F_y(P)) \neq 0$.
(nonsingular pt)

$\Rightarrow$ tangent line to F at P

$F_x(P)(x-a) + F_y(P)(y-b) = 0$

②

$P=$ multiple point of $F \overset{\text{def}}{\iff}$ otherwise
(singular point)

↖ 更精细的研究

- $P=(0,0)$ $F=\underset{\neq 0}{F_m}+F_{m+1}+\cdots+F_n$ ($F_i=$ form of deg i, $m\le n$)

$m_P(F):=m$ multiplicity of F at $P=(0,0)$

1° $P\in F \iff m_P(F)>0$

2° $P=$ simple point $\iff m_P(F)=1$
$\Rightarrow F_1=$ tangent line.

3° $P=$ double point $\overset{\text{def}}{\iff} m_P(F)=2$

4° $P=$ triple point $\overset{\text{def}}{\iff} m_P(F)=3$

$\vdots$

$$F_m=\prod L_i^{r_i}$$

$L_i=$ tangent lines to F at $P=(0,0)$.
$r_i=$ multiplicity of the tangent line L_i

Fact: $F=\prod \underset{\text{irr}}{F_i}^{e_i}$ Then

1) $m_P(F):=\sum e_i\, m_P(F_i)$

2) If L is tangent line to F_i with multiplicity r_i, then L is tangent line to F with multiplicity $\sum e_i r_i$.

when P simple?

$P \in F$ is simple $\Leftrightarrow$ $\exists ! i$ s.t. $P \in F_i$ &

P simple pt of F_i, F_i simple comp. of F.

extend definitions to a point $P=(a,b) \neq (0,0)$. Consider the linear translation $T(x,y) = (x+a, y+b)$. Then

$$F^T = F(x+a, y+b).$$

Using. F^T to define $m_P(F)$, tangent lines to F at P.

multiplicity of the tangent

§ 3.2. multiplicities and local rings

F = irr. plane curve, $P \in F$ find multiplicity of P on F via $\mathcal{O}_P(F)$.

$\forall G \in k[X,Y]$. $g := G \bmod (F) \in \Gamma(F) = k[X,Y]/(F)$.

Thm. F = irr. curve, $P \in F$.

$L = aX + bY + c$ through P not tangent to F at P then

(1) $m_P(F) = \dim_k \left(m_P(F)^n / m_P(F)^{n+1} \right)$ $\underline{n \gg 0}$

In particular, $m_P(F)$ depends only on $\mathcal{O}_P(F)$.

(2) P = simple $\Leftrightarrow$ $\mathcal{O}_P(F)$ = DVR

(3) if P simple then $\ell = L \bmod (F) \in \mathcal{O}_P(F)$ is a uniformizing parameter